

Low multipole contributions to the gravitational self-force

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We calculate the unregularized monopole and dipole contributions to the self-force acting on a particle of small mass in a circular orbit around a Schwarzschild black hole. From a self-force point of view, these non-radiating modes are as important as the radiating modes with $l \geq 2$. In fact, we demonstrate how the dipole self-force contributes to the dynamics even at the Newtonian level.

The self-acceleration of a particle is an inherently gauge-dependent concept, but the Lorenz gauge is often preferred because of its hyperbolic wave operator. Our results are in the Lorenz gauge and are also obtained in closed form, except for the even-parity dipole case where we formulate and implement a numerical approach.

I. INTRODUCTION

The capture of solar-mass compact objects by massive black holes residing in galactic centers has been identified as one of the most promising sources of gravitational waves for the Laser Interferometer Space Antenna [1]. The need for accurate templates for signal detection and source identification is currently motivating an intense effort from many workers to determine the motion of a relativistic two-body system in the small mass-ratio limit, without relying on slow-motion or weak-field approximations; for a review, see Ref. [2]. The work presented in this paper is part of this larger effort.

Gravitational self-force and the MiSaTaQuWa equations of motion

Consider a small body of mass m in orbit around a much larger black hole of mass M . In the test-mass limit ($m \rightarrow 0$) the motion of the small body is known to follow a geodesic in the spacetime geometry of the large black hole [3, 4, 5, 6, 7]. But as the mass of the smaller object is allowed to increase, deviations from geodesic motion become noticeable; these are associated with important physical effects such as radiation reaction and finite-mass (conservative) corrections to the orbital motion. In a sense, the motion is now geodesic in the perturbed spacetime that contains both the black hole and the orbiting body. If $g_{\alpha\beta}$ denotes the unperturbed metric of the central black hole, and if $h_{\alpha\beta}$ denotes the perturbation produced by the orbiting body, then the motion is formally geodesic in the perturbed metric $g_{\alpha\beta} + h_{\alpha\beta}$. When viewed from the background spacetime, the motion appears to be accelerated, and the agent that produces the acceleration is a gravitational self-force acting on the particle.

To turn these considerations into concrete equations of motion, it is desirable to formulate an approximation in which the details of the small body's internal structure

have a negligible influence on the body's orbital motion. In this approximation the body is modeled as a point particle possessing mass but no higher multipole moments, and its motion is fully described in terms of a world line γ . But to formulate equations of motion for this world line becomes problematic, as the field $h_{\alpha\beta}$ produced by a point particle necessarily diverges at the position of the particle. This means that an affine connection cannot be defined on the world line, and that the statement “the particle follows a geodesic of the perturbed spacetime” does not make immediate sense.

The task of regularizing $h_{\alpha\beta}$ near the world line and formulating meaningful equations of motion for the point particle was undertaken by Mino, Sasaki, and Tanaka [8], and also by Quinn and Wald [9]. An interesting reformulation of this work was recently given by Detweiler and Whiting [10], who showed that the perturbation can be uniquely decomposed into a symmetric-singular field $h_{\alpha\beta}^S$, and a regular-radiative field $h_{\alpha\beta}^R$; the full (retarded) perturbation is then $h_{\alpha\beta} = h_{\alpha\beta}^S + h_{\alpha\beta}^R$. Detweiler and Whiting were able to establish that while $h_{\alpha\beta}^S$ reproduces the singularity structure of the metric perturbation, it exerts no force on the point particle; the gravitational self-force is then produced entirely by $h_{\alpha\beta}^R$, which is a homogeneous, regular, smooth field in a neighbourhood of the world line.

The MiSaTaQuWa equations of motion [8, 9], in the Detweiler-Whiting formulation [10], take the following form. Let $z^\mu(\tau)$ be parametric relations that describe the particle's world line γ , with τ denoting proper time in the background spacetime of the central black hole. Let $u^\mu = dz^\mu/d\tau$ be the particle's four-velocity, normalized with respect to the unperturbed metric: $g_{\mu\nu}u^\mu u^\nu = -1$. Let $D/d\tau$ denote covariant differentiation along the world line, defined with respect to a connection compatible with $g_{\alpha\beta}$. Then the particle's equations of motion are

$$\begin{aligned} \frac{Du^\mu}{d\tau} &= a^\mu[h^R] \\ &\equiv -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu)(2h_{\nu\lambda;\rho}^R - h_{\lambda\rho;\nu}^R)u^\lambda u^\rho \end{aligned} \quad (1.1)$$

where a semicolon indicates covariant differentiation with respect to the background connection. The right-hand side of Eq. (1.1) is the gravitational self-acceleration of the point particle; multiplying by m would give the gravitational self-force. Equation (1.1) is equivalent to the statement that the particle moves on a geodesic in a spacetime with metric $g_{\alpha\beta} + h_{\alpha\beta}^R$, but the description of the world line refers to the background spacetime. The right-hand side of Eq. (1.1) is of order m , and the gravitational self-acceleration is therefore $O(m)$; it vanishes in the test-mass limit and the motion becomes geodesic (in the background spacetime).

The decomposition of $h_{\alpha\beta}$ into singular “S” and radiative “R” fields relies on a specific choice of gauge for the metric perturbation, which must satisfy the Lorenz gauge condition

$$\nabla_\beta \left(h^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} h_{\gamma\delta} \right) = 0. \quad (1.2)$$

This choice ensures that $h_{\alpha\beta}$ satisfies a (hyperbolic) wave equation and that the correct, retarded solution can be identified. The singularity structure of the perturbation near the world line can then be determined by a local analysis (see, for example, Ref. [2]), and $h_{\alpha\beta}^S$ is constructed without ambiguity so that it exerts no force on the particle. The regular field $h_{\alpha\beta}^R$ is then the difference between the retarded solution and the locally-constructed singular field; this satisfies a homogeneous version of the wave equation satisfied by the full metric perturbation, and the metric $g_{\alpha\beta} + h_{\alpha\beta}^R$ is a solution to the linearized Einstein field equations in vacuum.

The Lorenz gauge therefore presents itself as a preferred gauge for this problem, and it has been shown that in the first post-Newtonian approximation, Eq. (1.1) agrees with the standard Einstein-Infeld-Hoffmann equations of motion in a common domain of validity [11]. But it is important to note that the equations of motion of Eq. (1.1) are not gauge invariant [12]: different gauge choices will lead to different results.

Self-acceleration by mode sums

A concrete evaluation of Eq. (1.1) is challenging and involves a large number of steps; for this discussion we consider the case of a particle orbiting a Kerr black hole.

The first sequence of steps are concerned with the computation of the metric perturbation $h_{\alpha\beta}$ produced by a point particle moving on a specified geodesic of the Kerr spacetime. A method for doing this was elaborated by Lousto and Whiting [13] and Ori [14], building on the pioneering work of Teukolsky [15], Chrzanowski [16], and Wald [17]. The procedure consists of (i) solving the Teukolsky equation for one of the Newman-Penrose quantities ψ_0 and ψ_4 (complex components of the Weyl tensor) produced by the point particle; (ii) obtaining from ψ_0 or ψ_4 a related (Hertz) potential Ψ by integrating an ordinary differential equation; (iii) applying to Ψ a number

of differential operators to obtain the metric perturbation in a radiation gauge that differs from the Lorenz gauge; and (iv) performing a gauge transformation from the radiation gauge to the Lorenz gauge.

It is well known that the Teukolsky equation can be separated when ψ_0 or ψ_4 is expressed as a multipole expansion, summing over modes with (spheroidal-harmonic) indices l and m . In fact, the procedure outlined above relies heavily on this mode decomposition, and the metric perturbation returned at the end of the procedure is also expressed as a sum over modes $h_{\alpha\beta}^l$. (For each l , m ranges from $-l$ to l , and summation of m over this range is henceforth understood.) From these, mode contributions to the self-acceleration can be computed: $a^\mu[h_l]$ is obtained from Eq. (1.1) by substituting $h_{\alpha\beta}^l$ in place of $h_{\alpha\beta}^R$. These mode contributions do not diverge on the world line, but $a^\mu[h_l]$ is discontinuous at the radial position of the orbit. The sum over modes, on the other hand, does not converge, because the “bare” acceleration (constructed from the retarded field $h_{\alpha\beta}$) is formally infinite.

The next sequence of steps is concerned with the regularization of each $a^\mu[h_l]$ by removing the contribution from $h_{\alpha\beta}^S$ [18, 19, 20, 21, 22, 23]. The singular field can be constructed locally in a neighbourhood of the particle, and then decomposed into modes of multipole order l . This gives rise to modes $a^\mu[h_l^S]$ for the singular part of the self-acceleration; these are also finite and discontinuous, and their sum over l also diverges. But the true modes $a^\mu[h_l^R] = a^\mu[h_l] - a^\mu[h_l^S]$ of the self-acceleration are continuous at the radial position of the orbit, and their sum does converge to the particle’s acceleration. (It should be noted that obtaining a mode decomposition of the singular field involves providing an extension of $h_{\alpha\beta}^S$ on a sphere of constant radial coordinate, and then integrating over the angular coordinates. The arbitrariness of the extension introduces ambiguities in each $a^\mu[h_l^S]$, but the ambiguity disappears after summing over l .)

The gravitational self-acceleration is thus obtained by first computing $a^\mu[h_l]$ from the metric perturbation derived from ψ_0 or ψ_4 , then computing the counterterms $a^\mu[h_l^S]$ by mode-decomposing the singular field, and finally summing over all $a^\mu[h_l^R] = a^\mu[h_l] - a^\mu[h_l^S]$. This procedure is lengthy and involved, and thus far it has not been brought to completion, except for the special case of a particle falling radially toward a nonrotating black hole [24]. In this regard it should be noted that replacing the central Kerr black hole by a Schwarzschild black hole simplifies the task considerably. In particular, because there exists a practical and well-developed formalism to describe the metric perturbations of a Schwarzschild spacetime [25, 26, 27, 28, 29], there is no necessity to rely on the Teukolsky formalism and the complicated reconstruction of the metric variables.

Low multipoles — this work

The procedure described above is not complete. The reason is that the metric perturbations $h_{\alpha\beta}^l$ that can be recovered from ψ_0 or ψ_4 do not by themselves sum up to the complete gravitational perturbation produced by the moving particle. Missing are the perturbations derived from the other Newman-Penrose quantities: ψ_1 , ψ_2 , and ψ_3 . While ψ_1 and ψ_3 can always be set to zero by an appropriate choice of null tetrad, ψ_2 contains such important physical information as the shifts in mass and angular-momentum parameters produced by the particle [30]. Because the mode decompositions of ψ_0 and ψ_4 start at $l = 2$, we might colloquially say that what is missing from the above procedure are the “ $l = 0$ and $l = 1$ ” components of the metric perturbations. It is not currently known how the procedure can be completed so as to incorporate *all components* of the metric perturbations.

In this paper we consider the contribution of these low multipoles ($l = 0$ and $l = 1$) to the gravitational self-acceleration. To make progress we shall take the central black hole to be nonrotating, and the metric of the background spacetime to be a Schwarzschild solution. This simplification allows us to use the robust formalism of gravitational perturbations of the Schwarzschild spacetime [25, 26, 27, 28, 29], and more importantly, to define precisely what is meant by the “ $l = 0$ and $l = 1$ ” modes of the perturbation field. In this context the associations between the $l = 0$ mode and a shift of mass parameter, the odd-parity $l = 1$ mode and a shift of angular-momentum parameter, and the reduction of the even-parity $l = 1$ mode to a gauge transformation, were first established by Zerilli [26]. These associations are central to our discussion, and we believe that the results derived here will have a direct counterpart in the case of a Kerr black hole: The missing metric perturbations of the Kerr spacetime will be equivalent to our $l = 0$ and $l = 1$ perturbation modes in the limit where the black-hole angular momentum goes to zero.

To keep our discussion concrete and the mathematical complexities to a minimum, we calculate the $l = 0$ and $l = 1$ perturbation modes for the specific case of a particle moving on a circular orbit of radius R and angular velocity $\Omega = \sqrt{M/R^3}$. While finding solutions to the relevant perturbation equations can be a simple task when adopting a simple choice of gauge (as we shall see), we insist here, for reasons that were listed before, that the $l = 0$ and $l = 1$ perturbation modes should be calculated in the Lorenz gauge. This complicates the structure of the perturbation equations, and finding solutions is more challenging. We nevertheless are able to find exact analytical solutions for the cases $l = 0$ and $l = 1$ (odd parity). For even-parity $l = 1$, however, we have to rely on numerical methods for exact results, and a post-Newtonian approximation for analytical results.

In the remaining sections of the paper we calculate the

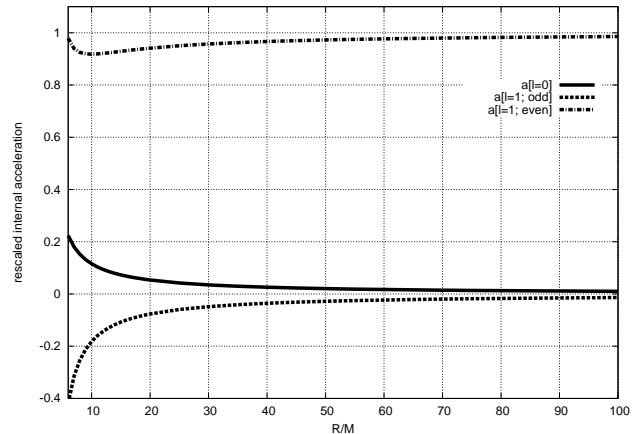


FIG. 1: Internal values of $a[l](R)$, rescaled by a common factor of $3m/R^2$. For $R \gg M$ we have the following asymptotic behaviors: $a_{<}[l = 0] \sim 3(m/R^2)(M/R)$, $a_{<}[l = 1; \text{odd}] \sim -4(m/R^2)(M/R)$, and $a_{<}[l = 1; \text{even}] \sim 3(m/R^2)$. An exact expression for $a_{<}[l = 0]$ appears in Eq. (3.15) below. An exact expression for $a_{<}[l = 1; \text{odd}]$ appears in Eq. (4.2). The values for $a_{<}[l = 1; \text{even}]$ are obtained from Eq. (5.55) and the results listed in Table I.

contributions

$$a^\mu[h_{l=0}], \quad a^\mu[h_{l=1;\text{odd parity}}], \quad a^\mu[h_{l=1;\text{even parity}}]$$

to the “bare” self-acceleration of a particle moving on a circular orbit around a Schwarzschild black hole. Our expressions are finite but discontinuous at the radial position of the particle: the answers obtained when approaching $r = R$ from the interior ($r < R$), and those obtained from exterior ($r > R$), do not match. We find in all cases that the contribution to the bare acceleration is purely radial: $a^\mu[h_l] = a^r[h_l]\delta_r^\mu$ for all three modes considered here. Moreover, in all cases the self-acceleration is conservative and does not contribute to the radiation reaction.

To keep the notation simple we shall set

$$\begin{aligned} a[l = 0] &\equiv a^r[h_{l=0}], \\ a[l = 1; \text{odd}] &\equiv a^r[h_{l=1;\text{odd parity}}], \\ a[l = 1; \text{even}] &\equiv a^r[h_{l=1;\text{even parity}}]. \end{aligned} \quad (1.3)$$

We display our results for $a[l](R)$ in two figures. In Fig. 1 we show the results as calculated from the orbit’s interior ($r \rightarrow R^-$) and in Fig. 2 we show our results as calculated from the orbit’s exterior ($r \rightarrow R^+$). These results do not have an immediate physical meaning. To produce meaning they must be included with higher-multipole contributions in a sum over all modes. Because there exists no procedure to uniquely remove the “S part” of the $l = 0$ and $l = 1$ perturbation modes (as was mentioned previously), we are not able here to produce expressions for the low-multipole contributions to the regularized self-acceleration.

Using purely analytical methods, Nakano, Sago and Sasaki [31] calculated the self-acceleration to first post-

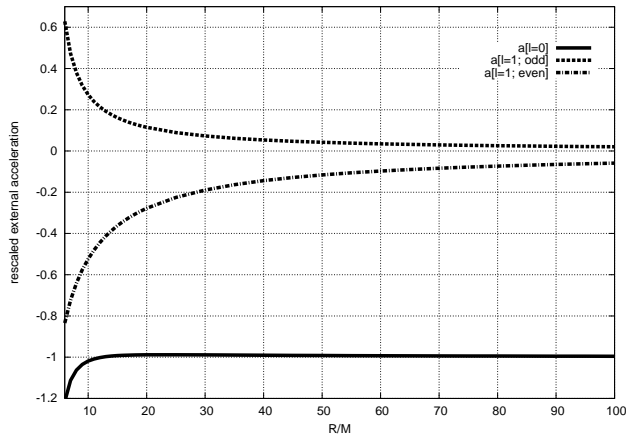


FIG. 2: External values of $a[l](R)$, rescaled by a common factor of m/R^2 . For $R \gg M$ we have the following asymptotic behaviors: $a_{>}[l=0] \sim -m/R^2$, $a_{<}[l=1;\text{odd}] \sim 2(m/R^2)(M/R)$, and $a_{<}[l=1;\text{even}] \sim -3\beta(m/R^2)(M/R)$, where $\beta \simeq 2$ is numerically estimated at the end of Sec. V. An exact expression for $a_{>}[l=0]$ appears in Eq. (3.16) below. An exact expression for $a_{>}[l=1;\text{odd}]$ appears in Eq. (4.3). The values for $a_{>}[l=1;\text{even}]$ are obtained from Eq. (5.55) and the results listed in Table I.

Newtonian order for circular orbits of the Schwarzschild geometry. For the even and odd parity $l=1$ modes, their results for the contribution to the “bare” self-acceleration agree with ours at the 1PN level, as expected. It appears that an extension of their methods to higher post-Newtonian orders might be substantially complicated by the difficulty caused by the even-parity $l=1$ perturbations, the case for which we have to rely on numerical methods. For the $l=0$ mode our results for the “bare” self-accelerations disagree with theirs at 1PN. We believe that the discrepancy is caused by the implementation of boundary conditions at the event horizon, and we discuss this matter in some detail in the Appendix.

Organization of this paper

In Sec. II we set the stage with a discussion of the gravitational self-force in Newtonian theory. This simple analogue to the relativistic problem sheds considerable light on the meaning of the self-acceleration and its decomposition into singular “S” and regular “R” fields. We show in particular that in Newtonian theory, the $l=1$ contribution to the self-acceleration is responsible for an important finite-mass correction to the particle’s angular velocity. We take this as a clear suggestion that in the relativistic problem, the low multipole contributions to the gravitational self-acceleration produce important physical effects.

In Sec. III we compute, in the Lorenz gauge, the $l=0$ gravitational perturbations produced by a particle in a circular orbit around a Schwarzschild black hole. These perturbations are associated with the change of mass pa-

rameter that occurs at $r=R$. We then calculate $a[l=0]$, the corresponding contribution to the self-acceleration.

In Sec. IV we do the same for the $l=1$, odd-parity perturbations. These are associated with the change of angular-momentum parameter that occurs across the orbit, and they give rise to the contribution $a[l=1;\text{odd}]$ to the self-acceleration.

In Sec. V we consider the $l=1$, even-parity gravitational perturbations, which are associated with the motion of the central black hole around the system’s center of mass. This calculation is considerably more involved than the others, because here the source of the perturbations is time dependent. Solving the vectorial wave equation that converts the perturbations from the Zerilli gauge to the Lorenz gauge requires numerical techniques, except when $R \gg M$ and we can rely on approximate analytical methods. In this section we obtain exact numerical results for $a[l=1;\text{even}]$, as well as approximate analytical results for $R \gg M$.

In Sec. VI we discuss our results and offer a number of concluding remarks.

II. NEWTONIAN SELF-ACCELERATION

In this section we consider a Newtonian system involving a large mass M and a much smaller mass m . The position of the small mass relative to the center of mass is described by the vector $\mathbf{R}(t)$, while the position of the larger mass is described by $\boldsymbol{\rho}(t)$. Taking the center of mass to be at the origin of the coordinate system, we have

$$m\mathbf{R} + M\boldsymbol{\rho} = \mathbf{0}. \quad (2.1)$$

We denote the position vector of an arbitrary field point by \mathbf{x} , and $r \equiv |\mathbf{x}|$ is its distance from the center of mass. We shall also use $R \equiv |\mathbf{R}|$ and $\rho \equiv |\boldsymbol{\rho}|$.

Test-mass description

We begin with a test-mass description of the situation, according to which the smaller mass moves in the gravitational field of the larger mass which is placed at the origin of the coordinate system. The background Newtonian potential is

$$\Phi_0(\mathbf{x}) = -\frac{M}{r} \quad (2.2)$$

and the background gravitational field is

$$\mathbf{g}_0 = -\nabla\Phi_0 = -\frac{M}{r^3}\mathbf{x}. \quad (2.3)$$

In this description, the smaller mass m moves according to $d^2\mathbf{R}/dt^2 = \mathbf{g}_0(\mathbf{x}=\mathbf{R})$. If the motion is circular, then m possesses a uniform angular velocity given by

$$\Omega_0^2 = \frac{M}{R^3}, \quad (2.4)$$

where R is the orbital radius. These results are in close analogy with a relativistic description in which the smaller mass is taken to move on a geodesic of the background spacetime, in a test-mass approximation.

Beyond the test-mass description: singular “S” and regular “R” perturbations of the Newtonian potential

We next improve our description by incorporating the gravitational effects produced by the smaller mass. The exact Newtonian potential is

$$\Phi(\mathbf{x}) = -\frac{M}{|\mathbf{x} - \boldsymbol{\rho}|} - \frac{m}{|\mathbf{x} - \mathbf{R}|}, \quad (2.5)$$

and for $m \ll M$ this can be expressed as $\Phi(\mathbf{x}) = \Phi_0(\mathbf{x}) + \delta\Phi(\mathbf{x})$, with a perturbation given by

$$\delta\Phi(\mathbf{x}) = -\frac{M}{|\mathbf{x} - \boldsymbol{\rho}|} + \frac{M}{r} - \frac{m}{|\mathbf{x} - \mathbf{R}|}. \quad (2.6)$$

This gives rise to a field perturbation $\delta\mathbf{g}$ that exerts a force on the smaller mass. This is the particle’s “bare” self-acceleration, and the correspondence with the relativistic problem is clear.

An examination of Eq. (2.6) reveals that the last term on the right-hand side diverges at the position of the smaller mass. But since the gravitational field produced by this term is isotropic around $\mathbf{R}(t)$, we know that this field will exert no force on the particle. We conclude that the last term can be identified with the singular “S” part of the perturbation,

$$\Phi_S(\mathbf{x}) = -\frac{m}{|\mathbf{x} - \mathbf{R}|}, \quad (2.7)$$

and that the remainder makes up the regular “R” field,

$$\Phi_R(\mathbf{x}) = -\frac{M}{|\mathbf{x} - \boldsymbol{\rho}|} + \frac{M}{r}. \quad (2.8)$$

The full perturbation is then given by $\delta\Phi(\mathbf{x}) = \Phi_S(\mathbf{x}) + \Phi_R(\mathbf{x})$, and only the “R potential” affects the motion of the smaller mass. Once more the correspondence with the relativistic problem is clear.

It is easy to check that to first order in m/M , Eq. (2.8) simplifies to

$$\Phi_R(\mathbf{x}) = m \frac{\mathbf{R} \cdot \mathbf{x}}{r^3}; \quad (2.9)$$

this simplification occurs because thanks to Eq. (2.1), $\boldsymbol{\rho}$ is formally of order $m/M \ll 1$. The regular “R” part of the field perturbation is then

$$\mathbf{g}_R(\mathbf{x}) = m \frac{3(\mathbf{R} \cdot \mathbf{x})\mathbf{x} - r^2\mathbf{R}}{r^5}, \quad (2.10)$$

and evaluating this at the particle’s position yields a correction to the background field $\mathbf{g}_0(\mathbf{x} = \mathbf{R}) = -M\mathbf{R}/R^3$

given by $\mathbf{g}_R(\mathbf{x} = \mathbf{R}) = 2m\mathbf{R}/R^3$; the force still points in the radial direction but the active mass has been shifted from M to $M - 2m$. For circular motion the angular velocity becomes

$$\Omega^2 = \frac{M - 2m}{R^3}. \quad (2.11)$$

This can be cast in a more recognizable form if we express the angular velocity in terms of the total separation $s \equiv R + \rho = (1 + m/M)R$ between the two masses. To first order in m/M we obtain $\Omega^2 = (M + m)/s^3$, which is just the usual form of Kepler’s third law. The regular part of the field perturbation is therefore responsible for the finite-mass correction to the angular velocity.

Multipole decomposition of the perturbations

We now examine the low-multipole content of $\delta\Phi(\mathbf{x})$, $\Phi_S(\mathbf{x})$, and $\Phi_R(\mathbf{x})$.

It is evident from Eq. (2.9) that $\Phi_R(\mathbf{x})$ possesses a pure dipolar form, and its multipole decomposition therefore involves a single term at $l = 1$. As we have seen, this dipole potential is responsible for an important finite-mass correction to the orbital frequency. We take this as a clear indication that in the relativistic context, the $l = 1$ contribution to the metric perturbations produces important physical effects that should not be ignored. Since there is no analogue in Newtonian theory to the odd parity metric perturbations, this statement might be restricted to the $l = 1$, even parity perturbations of the Schwarzschild spacetime.

While $\Phi_R(\mathbf{x})$ possesses only a dipole component, the same is not true of $\Phi_S(\mathbf{x})$ and $\delta\Phi(\mathbf{x})$. Their monopole components are given by

$$\Phi_S^{l=0}(\mathbf{x}) = \delta\Phi^{l=0}(\mathbf{x}) = \begin{cases} -m/R & r < R, \\ -m/r & r > R, \end{cases} \quad (2.12)$$

and this gives rise to a monopole field perturbation

$$\delta\mathbf{g}^{l=0} = \begin{cases} 0 & r < R, \\ -m\mathbf{x}/r^3 & r > R. \end{cases} \quad (2.13)$$

This is discontinuous at $\mathbf{x} = \mathbf{R}$: the field is zero when the limit is taken from the inside, and equal to $-m\mathbf{R}/R^3$ when taken from the outside. The jump in the monopole field perturbation is given by

$$\begin{aligned} [\delta\mathbf{g}^{l=0}] &\equiv \delta\mathbf{g}^{l=0}(\mathbf{x} = \mathbf{R}) \Big|_{\text{outside}} - \delta\mathbf{g}^{l=0}(\mathbf{x} = \mathbf{R}) \Big|_{\text{inside}} \\ &= -\frac{m}{R^3}\mathbf{R}. \end{aligned} \quad (2.14)$$

These results, which could be described as the Newtonian “bare” self-acceleration for $l = 0$, will be recovered as limits of our exact relativistic expressions in Sec. III.

The dipole component of the singular potential is calculated to be

$$\Phi_S^{l=1}(\mathbf{x}) = \begin{cases} -m(\mathbf{R} \cdot \mathbf{x})/R^3 & r < R, \\ -m(\mathbf{R} \cdot \mathbf{x})/r^3 & r > R, \end{cases} \quad (2.15)$$

and adding this to Eq. (2.9) we find

$$\delta\Phi^{l=1}(\mathbf{x}) = m(\mathbf{R} \cdot \mathbf{x}) \left(\frac{1}{r^3} - \frac{1}{R^3} \right) \quad (r < R) \quad (2.16)$$

and $\delta\Phi^{l=1}(\mathbf{x}) = 0$ for $r > R$. This gives rise to the field perturbation

$$\delta\mathbf{g}^{l=1} = m \left[\frac{3(\mathbf{R} \cdot \mathbf{x})\mathbf{x} - r^2\mathbf{R}}{r^5} + \frac{\mathbf{R}}{R^3} \right] \quad (r < R) \quad (2.17)$$

and $\delta\mathbf{g}^{l=1} = 0$ for $r > R$. The dipole field also is discontinuous at $\mathbf{x} = \mathbf{R}$: it is zero when the limit is taken from the outside, and equal to $3m\mathbf{R}/R^3$ when taken from the inside. Its jump, defined as in Eq. (2.14), is given by

$$[\delta\mathbf{g}^{l=0}] = -\frac{3m}{R^3}\mathbf{R}. \quad (2.18)$$

These results, which could be described as the Newtonian “bare” self-acceleration for $l = 1$, will also be recovered as limits of our exact relativistic expressions in Sec. V.

III. MONOPOLE GRAVITATIONAL PERTURBATIONS

Our task in this section is to calculate the $l = 0$ metric perturbations of the Schwarzschild spacetime produced by a particle of mass m in circular orbit at a radius R . We shall also calculate the associated contribution to the self-acceleration, $a[l = 0]$ as defined by Eq. (1.3). We need the perturbations in the Lorenz gauge, and our strategy will be to obtain them first in the simpler Zerilli gauge, and then look for a transformation to the Lorenz gauge.

Perturbations in the Zerilli gauge

The monopole perturbations produced by a point particle in arbitrary motion around a Schwarzschild black hole were first computed by Zerilli [26]. With his specific choice of gauge for circular motion, the metric perturbations are

$$h_{tt}^Z = 2m\tilde{E} \left(\frac{1}{r} - \frac{f}{R - 2M} \right) \Theta(r - R) \quad (3.1)$$

and

$$h_{rr}^Z = \frac{2m\tilde{E}}{rf^2} \Theta(r - R), \quad (3.2)$$

where $f = 1 - 2M/r$, $\tilde{E} = (1 - 2M/R)(1 - 3M/R)^{-1/2}$ is the particle’s energy per unit rest mass, and $\Theta(r - R)$ is the Heaviside step function. It is easy to check that for $r > R$, $g_{\alpha\beta} + h_{\alpha\beta}^Z$ is another Schwarzschild metric with mass parameter $M + m\tilde{E}$. The perturbation therefore describes the sudden shift in mass parameter that occurs at $r = R$.

Transformation to the Lorenz gauge

The metric perturbation of Eqs. (3.1) and (3.2) does not satisfy the Lorenz gauge condition of Eq. (1.2). We therefore seek a vector field ξ^α that generates a transformation from the Zerilli gauge to the Lorenz gauge. This vector must possess only an $l = 0$ component, and so it must be of the form $\xi_\alpha = [0, \xi(r), 0, 0]$. As the perturbation is static, there is no need to include a component in the time direction (this point is elaborated in the Appendix), nor a time dependence in the radial component.

To find this vector we express the Lorenz-gauge metric perturbations in the standard Regge-Wheeler [25] form

$$\begin{aligned} h_{tt} &= fH_0(r), \\ h_{rr} &= H_2(r)/f, \\ h_{AB} &= r^2\Omega_{AB}K(r), \end{aligned} \quad (3.3)$$

where the upper-case latin indices run over the angular coordinates θ and ϕ , and $\Omega_{AB} = \text{diag}(1, \sin^2\theta)$ is the metric of the unit two-sphere. We have set $h_{tr} = H_1(r) = 0$ on the grounds that the perturbation must be static. A similar notation can be used to express the Zerilli-gauge perturbations, and we have $K^Z = 0$ while H_0^Z and H_2^Z are nonzero. The gauge transformation is given by $h_{\alpha\beta} = h_{\alpha\beta}^Z - 2\xi_{(\alpha;\beta)}$ and this translates to

$$\begin{aligned} H_0 &= H_0^Z + \frac{2M}{r^2}\xi, \\ H_2 &= H_2^Z - 2f\xi' - \frac{2M}{r^2}\xi, \\ K &= -\frac{2f}{r}\xi, \end{aligned} \quad (3.4)$$

where a prime indicates differentiation with respect to r . The new perturbation will satisfy the Lorenz gauge condition if

$$f(H_0' + H_2' - 2K') + \frac{2M}{r^2}H_0 + \frac{2(2r - 3M)}{r^2}H_2 - \frac{4f}{r}K = 0. \quad (3.5)$$

Using Eqs. (3.1), (3.2), and (3.4), this becomes an ordinary differential equation for $\xi(r)$:

$$\begin{aligned} f\xi'' + \frac{2}{r}\xi' - \frac{2f}{r^2}\xi &= \frac{m\tilde{E}}{R - 2M}\delta(r - R) \\ &+ \frac{2m\tilde{E}}{r^2f} \frac{R - 3M}{R - 2M} \Theta(r - R). \end{aligned} \quad (3.6)$$

Our task is now to find a solution to this equation.

The function $\xi(r)$ can be expressed as a superposition of interior and exterior solutions,

$$\xi(r) = \xi_{<}(r)\Theta(R - r) + \xi_{>}(r)\Theta(r - R). \quad (3.7)$$

The interior solution $\xi_{<}(r)$ satisfies the homogeneous version of Eq. (3.6), while the exterior solution $\xi_{>}(r)$ satisfies Eq. (3.6) with $\delta(r - R)$ set equal to zero and $\Theta(r - R)$

set equal to 1. The solutions must comply with the jump conditions

$$[\xi] = 0, \quad [\xi'] = \frac{m\tilde{E}R}{(R-2M)^2}, \quad (3.8)$$

where $[\xi] \equiv \xi_>(r=R) - \xi_<(r=R)$ and a similar definition holds for $[\xi']$. Equations (3.8) and (3.4) imply that in the Lorenz gauge, the metric perturbations are continuous at $r=R$: $[H_0] = [H_2] = [K] = 0$.

The interior solution is a linear superposition of the two independent solutions $\xi_1 = [r(r-2M)]^{-1}$ and $\xi_2 = r^2/(r-2M)$. Regularity at the event horizon requires that ξ be well behaved in the limit $r \rightarrow 2M$. We must therefore choose

$$\xi_<(r) = a \frac{r^2 + 2Mr + 4M^2}{r}, \quad (3.9)$$

where a is a constant that will be determined by the jump conditions. The exterior solution is a linear superposition of ξ_1 , ξ_2 , and the particular solution $\xi_p = -m\tilde{E}(R-3M)(R-2M)^{-1}\Gamma(r)$, where

$$\Gamma(r) = -\left\{ [9Mr(r-2M)]^{-1} [3r^3 \ln(1-2M/r) - 3Mr^2 - 12M^2r + 44M^3 - 24M^3 \ln[r/(2M) - 1]] \right\}. \quad (3.10)$$

Because $\xi_1 \sim 1/r^2$, $\xi_2 \sim r$, and $\Gamma \sim 1$ when $r \rightarrow \infty$, proper asymptotic behavior requires that we discard ξ_2 from the exterior solution. We then have

$$\xi_>(r) = b \frac{M^3}{r(r-2M)} - m\tilde{E} \frac{R-3M}{R-2M} \Gamma(r), \quad (3.11)$$

where b is a constant that will be determined by the jump conditions.

The gauge vector is now fully determined: The interior solution is given by Eq. (3.9) and the exterior solution by Eq. (3.11) with the function $\Gamma(r)$ displayed in Eq. (3.10). The complete gauge vector field is then constructed as in Eq. (3.7), and the constants a and b are determined by the jump conditions of Eq. (3.8). This is sufficient information to calculate the Lorenz-gauge metric perturbations with the help of Eqs. (3.3) and (3.4). Because the resulting expressions are moderately lengthy, we shall not display these results here, but proceed instead with the calculation of the self-acceleration.

Before moving on we wish to call attention to the fact that in the foregoing manipulations, the requirements of staticity, regularity at the event horizon, and regularity at infinity have allowed us to construct a *unique solution* to the perturbation equations in the Lorenz gauge. This conclusion is elaborated in the Appendix.

Monopole contribution to the self-acceleration

The self-acceleration produced by the $l=0$ perturbations can be expressed as a sum of two terms,

$$a[l=0] = a[l=0; \text{Zerilli}] + a[l=0; \text{gauge}], \quad (3.12)$$

where $a[l=0; \text{Zerilli}]$ is the radial component of the acceleration vector constructed as in Eq. (1.1) but by replacing $h_{\alpha\beta}^R$ with $h_{\alpha\beta}^Z$, while $a[l=0; \text{gauge}]$ is constructed from $h_{\alpha\beta}^{\text{gauge}} = -2\xi_{(\alpha;\beta)}$. The calculation involves the particle's velocity vector $u^\mu = (1-3M/R)^{-1/2}[1, 0, 0, \Omega]$, and at the end $h_{\alpha\beta;\gamma}$ must be evaluated at the position of the particle ($r=R$, $\theta=\pi/2$, and $\phi=\Omega t$), either from the orbit's interior ($r < R$) or from its exterior ($r > R$). This leads to two different values for the acceleration, $a_<$ and $a_>$, respectively. Such a discontinuity was encountered before in a Newtonian context — refer back to Eq. (2.13).

The external value of the Zerilli acceleration is given by

$$a_>[l=0; \text{Zerilli}] = -\frac{m\tilde{E}}{R(R-3M)}, \quad (3.13)$$

while the internal value is zero: $a_<[l=0; \text{Zerilli}] = 0$. The gauge acceleration, on the other hand, is found to be

$$a[l=0; \text{gauge}] = -\frac{3M(R-2M)^2}{R^4(R-3M)}\xi(R),$$

and by virtue of Eq. (3.8), the internal and external values are equal. The gauge vector can most simply be evaluated from the orbit's interior, and Eq. (3.9) gives $\xi(R) = a(R^2 + 2MR + 4M^2)/R$. But the jump conditions imply $a = \frac{1}{3}(m\tilde{E}/M)[(R-3M)\ln(1-2M/R) - M]/(R-2M)$, and altogether we obtain

$$a[l=0; \text{gauge}] = m\tilde{E} \frac{(R-2M)(R^2 + 2MR + 4M^2)}{R^5} \times \left[\frac{M}{R-3M} - \ln\left(1 - \frac{2M}{R}\right) \right]. \quad (3.14)$$

From Eqs. (3.12)–(3.14) we arrive at our final results. The internal value for the $l=0$ self-acceleration is

$$a_<[l=0] = m\tilde{E} \frac{(R-2M)(R^2 + 2MR + 4M^2)}{R^5} \times \left[\frac{M}{R-3M} - \ln\left(1 - \frac{2M}{R}\right) \right], \quad (3.15)$$

while the external value is

$$a_>[l=0] = -m\tilde{E} \frac{R^4 - MR^3 + 8M^4}{R^5(R-3M)} - m\tilde{E} \frac{(R-2M)(R^2 + 2MR + 4M^2)}{R^5} \times \ln\left(1 - \frac{2M}{R}\right). \quad (3.16)$$

When $R \gg M$ these expressions simplify to $a_<[l=0] \sim 3mM/R^3$ and $a_>[l=0] \sim -m/R^2 + mM/(2R^3)$; the internal value is smaller than the external value by a factor of order $M/R \ll 1$. These limiting expressions are compatible with the Newtonian results displayed in

Eq. (2.13). They differ, however, from the results of Nakano, Sago, and Sasaki [31], which are displayed in their Eq. (E19) — our expressions are smaller than theirs by a term $4mM/R^3$. This discrepancy is explained in the Appendix. Equations (3.15) and (3.16) were used to generate the curves shown in Figs. 1 and 2.

IV. DIPOLE, ODD-PARITY GRAVITATIONAL PERTURBATIONS

In this section we calculate the $l = 1$, odd-parity metric perturbations of the Schwarzschild spacetime produced by a particle of mass m in circular orbit at a radius R . From these we shall derive their contribution to the self-acceleration, $a[l = 1; \text{odd}]$ as defined by Eq. (1.3). Here we shall find that the expressions provided by Zerilli already satisfy the Lorenz gauge condition.

The dipole, odd-parity perturbations produced by a point particle in arbitrary motion around a Schwarzschild black hole were first computed by Zerilli [26] and shown to be intimately related to the shift in angular-momentum parameter that occurs at the orbit. After specializing to circular motion in the equatorial plane, his results read

$$h_{t\phi} = -2m\tilde{L}\sin^2\theta \times \begin{cases} r^2/R^3 & r < R, \\ 1/r & r > R, \end{cases} \quad (4.1)$$

where $\tilde{L} = [MR/(1 - 3M/R)]^{1/2}$ is the particle's angular momentum per unit rest mass. For $r < R$, the metric $g_{\alpha\beta} + h_{\alpha\beta}$ differs from $g_{\alpha\beta}$ only by a gauge transformation — it is also a Schwarzschild metric with mass parameter M . For $r > R$, $g_{\alpha\beta} + h_{\alpha\beta}$ is a Kerr metric linearized with respect to the angular-momentum parameter $a \equiv (m/M)\tilde{L}$. The perturbation therefore describes the sudden shift in angular momentum that occurs at $r = R$.

It is easy to check that the perturbation of Eq. (4.1) satisfies the Lorenz gauge condition of Eq. (1.2). It is also easy to show that a (time-independent) gauge transformation within the class of Lorenz gauges would produce a pathological behavior of the perturbation at the event horizon. Equation (4.1) therefore gives us a unique solution to the perturbation equations in the Lorenz gauge.

A straightforward calculation then reveals that the internal value of the $l = 1$, odd-parity contribution to the self-acceleration is

$$a_{<}[l = 1; \text{odd}] = -\frac{4mM}{R^3} \frac{1 - 2M/R}{(1 - 3M/R)^{3/2}}, \quad (4.2)$$

while the external value is

$$a_{>}[l = 1; \text{odd}] = \frac{2mM}{R^3} \frac{1 - 2M/R}{(1 - 3M/R)^{3/2}}. \quad (4.3)$$

These results have no analogue in Newtonian theory. Equations (4.2) and (4.3) were used to generate the curves shown in Figs. 1 and 2.

V. DIPOLE, EVEN-PARITY GRAVITATIONAL PERTURBATIONS

Our task in this section is to calculate the $l = 1$, even-parity metric perturbations of the Schwarzschild spacetime produced by a particle of mass m in circular orbit at a radius R . We shall also calculate the associated contribution to the self-acceleration, $a[l = 1; \text{even}]$ as defined by Eq. (1.3). Once more we need the perturbations in the Lorenz gauge, and as in Sec. III our strategy will be to obtain them first in the simpler Zerilli gauge, and then look for a transformation to the Lorenz gauge. The solution to the wave equation satisfied by the gauge vector field will be obtained numerically and provided in tabulated form. It will also be obtained analytically in a post-Newtonian expansion in powers of M/R .

Perturbations in the Zerilli gauge

The dipole, even-parity perturbations produced by a point particle in arbitrary motion around a Schwarzschild black hole were first computed by Zerilli [26] in a simple choice of gauge. After specializing to circular motion, his results become

$$h_{tt}^Z = 2m\tilde{E} \frac{R - 2M}{r(r - 2M)} (1 - r^3\Omega^2/M) \times \sin\theta \cos(\phi - \Omega t)\Theta(r - R), \quad (5.1)$$

$$h_{tr}^Z = -6m\tilde{E}\Omega(R - 2M) \frac{r}{(r - 2M)^2} \times \sin\theta \sin(\phi - \Omega t)\Theta(r - R), \quad (5.2)$$

$$h_{rr}^Z = 6m\tilde{E}(R - 2M) \frac{r}{(r - 2M)^3} \times \sin\theta \cos(\phi - \Omega t)\Theta(r - R), \quad (5.3)$$

where $\Omega = \sqrt{M/R^3}$ is the particle's angular velocity and $\tilde{E} = (1 - 2M/R)(1 - 3M/R)^{-1/2}$ is its energy per unit mass. Here we see that the perturbations are time dependent, and this complicates considerably the task of finding the transformation to the Lorenz gauge. Equation (5.1) reveals that the Zerilli gauge is not asymptotically flat, since h_{tt}^Z grows linearly with r as $r \rightarrow \infty$. This indicates the fact that the metric $g_{\alpha\beta} + h_{\alpha\beta}^Z$ is expressed in a noninertial coordinate system anchored to the black hole instead of the system's center of mass. This statement will be elaborated below.

Perturbations in a singular gauge

The metric perturbations of Eqs. (5.1)–(5.3) do not satisfy the Lorenz gauge condition of Eq. (1.2). To transform to the Lorenz gauge we proceed in two steps. We shall first transform to a gauge in which the perturbation is zero everywhere, except at $r = R$ where it is singular. We shall then go from this singular gauge to the Lorenz gauge.

It is well known from Zerilli's work [26] that *in vacuum*, a dipole, even-parity perturbation can be completely removed by a gauge transformation. Such a perturbation, therefore, represents a coordinate transformation; and as we have already suggested, for $r > R$ the metric $g_{\alpha\beta} + h_{\alpha\beta}^Z$ is just a Schwarzschild solution expressed in a noninertial coordinate system. The perturbations of Eqs. (5.1)–(5.3), however, are not pure gauge because of the presence of the particle. They can be removed in the vacuum region outside of $r = R$, but the gauge transformation leaves something behind at $r = R$. The result of this transformation is $h_{\alpha\beta}^s$, the metric perturbation in what we shall call the *singular gauge*.

The gauge transformation that removes a dipole, even-parity perturbation in vacuum was constructed by Zerilli [26]. It is generated by a vector field ε^α , so that $h_{\alpha\beta}^Z = h_{\alpha\beta}^s - 2\varepsilon_{(\alpha;\beta)}$. For circular motion this is given by

$$\varepsilon_t = \frac{m\tilde{E}}{M} \Omega(R-2M) \frac{r^2}{r-2M} \times \sin\theta \sin(\phi - \Omega t) \Theta(r-R), \quad (5.4)$$

$$\varepsilon_r = -\frac{m\tilde{E}}{M} (R-2M) \frac{r^2}{(r-2M)^2} \times \sin\theta \cos(\phi - \Omega t) \Theta(r-R), \quad (5.5)$$

$$\varepsilon_\theta = -\frac{m\tilde{E}}{M} (R-2M) \frac{r^2}{r-2M} \times \cos\theta \cos(\phi - \Omega t) \Theta(r-R), \quad (5.6)$$

$$\varepsilon_\phi = \frac{m\tilde{E}}{M} (R-2M) \frac{r^2}{r-2M} \times \sin\theta \sin(\phi - \Omega t) \Theta(r-R). \quad (5.7)$$

The new metric perturbation is then

$$h_{tr}^s = -\frac{m\tilde{E}}{M} \Omega R^2 \sin\theta \sin(\phi - \Omega t) \delta(r-R), \quad (5.8)$$

$$h_{rr}^s = 2\frac{m\tilde{E}}{M} \frac{R^2}{R-2M} \sin\theta \cos(\phi - \Omega t) \delta(r-R), \quad (5.9)$$

$$h_{r\theta}^s = \frac{m\tilde{E}}{M} R^2 \cos\theta \cos(\phi - \Omega t) \delta(r-R), \quad (5.10)$$

$$h_{r\phi}^s = -\frac{m\tilde{E}}{M} R^2 \sin\theta \sin(\phi - \Omega t) \delta(r-R), \quad (5.11)$$

and we see that in the singular gauge, the metric perturbation is proportional to $\delta(r-R)$, which is produced by differentiation of the step function in ε_α . The gauge transformation therefore makes the perturbation zero everywhere in the vacuum region outside (and inside) $r = R$, but it contributes a singular term at the orbit. This illustrates the fact that the presence of matter prevents the Zerilli-gauge metric perturbation from being pure gauge.

Interpretation of the gauge transformation

The preceding discussion on coordinate systems can be clarified if we examine the asymptotic behavior of the

gauge vector field in the limit $r \rightarrow \infty$. In this limit we can seek a Newtonian interpretation of the results, and we shall see that in the original Zerilli gauge, the perturbed metric is that of a moving black hole. The following is patterned after a similar discussion produced by Zerilli [26].

The vector ε^α becomes asymptotically equal to b^α in the limit $r \rightarrow \infty$, where

$$\begin{aligned} b^t &= -\frac{m\tilde{E}}{M} (R-2M) \frac{\partial}{\partial t} r \sin\theta \cos(\phi - \Omega t), \\ b^r &= -\frac{m\tilde{E}}{M} (R-2M) \sin\theta \cos(\phi - \Omega t), \\ b^\theta &= -\frac{m\tilde{E}}{M} (R-2M) \frac{1}{r} \cos\theta \cos(\phi - \Omega t), \\ b^\phi &= \frac{m\tilde{E}}{M} (R-2M) \frac{1}{r \sin\theta} \sin(\phi - \Omega t). \end{aligned}$$

If we introduce asymptotic Cartesian coordinates $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r \cos\theta$, we have

$$\begin{aligned} b^t &= -\frac{m\tilde{E}}{M} (R-2M) \frac{\partial}{\partial t} (x \cos\Omega t + y \sin\Omega t), \\ b^x &= -\frac{m\tilde{E}}{M} (R-2M) \cos\Omega t, \\ b^y &= -\frac{m\tilde{E}}{M} (R-2M) \sin\Omega t, \\ b^z &= 0. \end{aligned}$$

To give a Newtonian interpretation to these results, let \mathbf{x} be the Zerilli coordinates of an arbitrary field point, let $\mathbf{R}(t) \equiv (R-2M)(\cos\Omega t, \sin\Omega t, 0)$ be the position vector of the orbiting particle, and express the preceding equations as

$$\mathbf{b}(t) = -\frac{m\tilde{E}}{M} \mathbf{R}(t), \quad b^t = \mathbf{x} \cdot \dot{\mathbf{b}}(t),$$

where an overdot indicates differentiation with respect to t . The coordinate transformation generated by b^α is then

$$\mathbf{x}_{\text{new}} = \mathbf{x} + \mathbf{b}(t), \quad t_{\text{new}} = t + \mathbf{x} \cdot \dot{\mathbf{b}}(t).$$

We can now explain that this transformation represents a translation from a noninertial reference frame attached to the black hole to an inertial frame attached to the center of mass. Please refer back to Sec. II for a definition of the notation employed here.

In the center-of-mass frame, the particle moves on a trajectory $\mathbf{R}_{\text{cm}}(t)$, and the black hole moves on a trajectory $\boldsymbol{\rho}_{\text{cm}}(t)$. In the black-hole frame we have $\mathbf{R}_{\text{bh}}(t) \equiv \mathbf{R}(t)$ and $\boldsymbol{\rho}_{\text{bh}}(t) \equiv \mathbf{0}$. The center-of-mass condition is $(m\tilde{E})\mathbf{R}_{\text{cm}} + M\boldsymbol{\rho}_{\text{cm}} = \mathbf{0}$, so we have $\boldsymbol{\rho}_{\text{cm}} = -(m\tilde{E}/M)\mathbf{R}_{\text{cm}}$. We also have $\mathbf{R} = \mathbf{R}_{\text{cm}} - \boldsymbol{\rho}_{\text{cm}} = (1 + m\tilde{E}/M)\mathbf{R}_{\text{cm}}$, so that $\mathbf{R}_{\text{cm}} = \mathbf{R} + O(m/M)$ and $\boldsymbol{\rho}_{\text{cm}} = -(m\tilde{E}/M)\mathbf{R} + O(m^2/M^2)$, or

$$\boldsymbol{\rho}_{\text{cm}}(t) = \mathbf{b}(t) + O(m^2/M^2).$$

The vector $\mathbf{b}(t)$ is therefore the position of the black hole relative to the center of mass, and the coordinate transformation is truly a translation from the moving frame of the black hole to the fixed reference frame of the center of mass.

Transformation to the Lorenz gauge

We now return to the task of transforming the metric perturbation from the singular gauge of Eqs. (5.8)–(5.11) to the Lorenz gauge. The gauge transformation is generated by a vector field ξ^α , such that

$$h_{\alpha\beta} = h_{\alpha\beta}^s - 2\xi_{(\alpha;\beta)} \quad (5.12)$$

is the Lorenz-gauge metric perturbation. For this to comply with Eq. (1.2), the vector field must satisfy the inhomogeneous wave equation

$$\square \xi^\alpha = S^\alpha, \quad (5.13)$$

where $\square = \nabla^\beta \nabla_\beta$ is the wave operator and

$$S^\alpha = \nabla_\beta \left(h_s^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} h_{\gamma\delta}^s \right) \quad (5.14)$$

is the source term. This is given explicitly by

$$S^t = \frac{m\tilde{E}}{M} \frac{\Omega R}{R-2M} \left[(3R-2M)\delta(r-R) + R(R-2M)\delta'(r-R) \right] \sin\theta \sin(\phi - \Omega t), \quad (5.15)$$

$$S^r = \frac{m\tilde{E}}{M} \frac{1}{R} \left[(2R-5M)\delta(r-R) + R(R-2M)\delta'(r-R) \right] \sin\theta \cos(\phi - \Omega t), \quad (5.16)$$

$$S^\theta = \frac{m\tilde{E}}{M} \frac{1}{R^2} \left[(3R-8M)\delta(r-R) + R(R-2M)\delta'(r-R) \right] \cos\theta \cos(\phi - \Omega t), \quad (5.17)$$

$$S^\phi = -\frac{m\tilde{E}}{M} \frac{1}{R^2 \sin^2\theta} \left[(3R-8M)\delta(r-R) + R(R-2M)\delta'(r-R) \right] \sin\theta \sin(\phi - \Omega t). \quad (5.18)$$

To arrive at these results we have invoked the distributional identity $g(r)\delta'(r-R) = g(R)\delta'(r-R) - g'(R)\delta(r-R)$, where $g(r)$ is any test function and $g'(r)$ its derivative with respect to r .

To solve Eq. (5.13) we decompose the vector ξ_α in even-parity spherical harmonics of degree $l = 1$. The form of the source term indicates that only terms with $m = \pm 1$ are needed, and we let

$$\xi_a(t, r, \theta^A) = \sum_{\pm} \xi_a^\pm(t, r) Y^\pm(\theta^A), \quad (5.19)$$

$$\xi_A(t, r, \theta^A) = \sum_{\pm} \xi^\pm(t, r) \partial_A Y^\pm(\theta^A). \quad (5.20)$$

Here, the lower-case latin index a refers to the t and r components of the vector field, while the upper-case index A refers to the angular components; we have set $\theta^A = (\theta, \phi)$ and $Y^\pm(\theta^A) \equiv Y_{l=1}^{m=\pm 1}(\theta^A) = \mp \sqrt{3/(8\pi)} \sin\theta e^{\pm i\phi}$.

The vector S^α can be decomposed in a similar way, and to simplify the form of the reduced wave equation we define the functions $A^\pm(r)$, $B^\pm(r)$, and $C^\pm(r)$ by the relations

$$\xi_t^\pm(t, r) = -\frac{1}{2} \sqrt{\frac{8\pi}{3}} \frac{m\tilde{E}}{M} i\Omega R \frac{A^\pm(r)}{r} e^{\mp i\Omega t}, \quad (5.21)$$

$$\xi_r^\pm(t, r) = \mp \frac{1}{2} \sqrt{\frac{8\pi}{3}} \frac{m\tilde{E}}{M} \frac{B^\pm(r)}{r-2M} e^{\mp i\Omega t}, \quad (5.22)$$

$$\xi^\pm(t, r) = \mp \frac{1}{2} \sqrt{\frac{8\pi}{3}} \frac{m\tilde{E}}{M} C^\pm(r) e^{\mp i\Omega t}. \quad (5.23)$$

With these definitions Eq. (5.13) becomes the following set of ordinary differential equations:

$$\frac{d^2 A^\pm}{dr^2} + \left[\frac{\Omega^2 r^2}{(r-2M)^2} - \frac{2}{r(r-2M)} \right] A^\pm - \frac{2M/R}{(r-2M)^2} B^\pm = \frac{2R^2}{R-2M} \delta(r-R) + R^2 \delta'(r-R), \quad (5.24)$$

$$\frac{d^2 B^\pm}{dr^2} + \left[\frac{\Omega^2 r^2}{(r-2M)^2} - \frac{4(r-M)}{r^2(r-2M)} \right] B^\pm + \frac{2\Omega^2 MR}{(r-2M)^2} A^\pm + \frac{4}{r^2} C^\pm = \frac{R(R-M)}{R-2M} \delta(r-R) + R^2 \delta'(r-R), \quad (5.25)$$

$$\frac{d^2 C^\pm}{dr^2} + \frac{2M}{r(r-2M)} \frac{dC^\pm}{dr} + \left[\frac{\Omega^2 r^2}{(r-2M)^2} - \frac{2}{r(r-2M)} \right] C^\pm + \frac{2}{r(r-2M)} B^\pm = R\delta(r-R) + R^2 \delta'(r-R). \quad (5.26)$$

The steps required to obtain the Lorenz-gauge metric

perturbation are therefore these: First, solve Eqs. (5.24)–

(5.26) for the functions $A^\pm(r)$, $B^\pm(r)$, and $C^\pm(r)$; second, insert the solutions into Eqs. (5.21)–(5.23), and these into Eqs. (5.19) and (5.20), to construct the gauge vector field ξ_α ; third, compute $h_{\alpha\beta}$ using Eq. (5.12).

Jump conditions and asymptotic behavior

The solutions to Eqs. (5.24)–(5.26) can be expressed as

$$A^\pm(r) = A_{<}^\pm(r)\Theta(R-r) + A_{>}^\pm(r)\Theta(r-R), \quad (5.27)$$

$$B^\pm(r) = B_{<}^\pm(r)\Theta(R-r) + B_{>}^\pm(r)\Theta(r-R), \quad (5.28)$$

$$C^\pm(r) = C_{<}^\pm(r)\Theta(R-r) + C_{>}^\pm(r)\Theta(r-R), \quad (5.29)$$

where the interior and exterior solutions satisfy the corresponding homogeneous equations. To account for the source terms, these functions must comply with the jump conditions

$$[A^\pm] = [B^\pm] = [C^\pm] = R^2 \quad (5.30)$$

and

$$\left[\frac{dA^\pm}{dr}\right] = \frac{2R^2}{R-2M}, \quad (5.31)$$

$$\left[\frac{dB^\pm}{dr}\right] = \frac{R(R-M)}{R-2M}, \quad (5.32)$$

$$\left[\frac{dC^\pm}{dr}\right] = \frac{R(R-4M)}{R-2M}, \quad (5.33)$$

where $[\psi] \equiv \psi_{>}(r=R) - \psi_{<}(r=R)$ for any function ψ of the radial coordinate.

Near the event horizon the interior functions can be expanded as

$$A_{<}^\pm(r) = e^{\mp i\Omega r^*} \sum_{n=0}^{\infty} a_n (r-2M)^n, \quad (5.34)$$

$$B_{<}^\pm(r) = e^{\mp i\Omega r^*} \sum_{n=0}^{\infty} b_n (r-2M)^n, \quad (5.35)$$

$$C_{<}^\pm(r) = e^{\mp i\Omega r^*} \sum_{n=0}^{\infty} c_n (r-2M)^n. \quad (5.36)$$

These forms ensure that the vector ξ_α satisfies ingoing-wave boundary conditions at the horizon, a necessary condition to obtain a *retarded solution* to Eq. (5.13). Substitution into Eqs. (5.24)–(5.26) provides recurrence relations consisting of three coupled expressions for the a_n , b_n , and c_n :

$$\begin{aligned} [2Mn(n-1) \mp 4i\Omega M^2(2n-1)]a_n - 4M^2R^{-1}b_n \\ = -[n(n-3) \mp 2i\Omega M(4n-5)]a_{n-1} \mp 2i\Omega(2-n)a_{n-2} + 2MR^{-1}b_{n-1}, \end{aligned} \quad (5.37)$$

$$\begin{aligned} 8\Omega^2 M^3 R a_n + [4M^2 n(n-1) \mp 8i\Omega M^3(2n-1)]b_n \\ = -8\Omega^2 M^2 R a_{n-1} - 2\Omega^2 M R a_{n-2} - [4M(n^2 - 3n + 1) \mp 8i\Omega M^2(3n-4)]b_{n-1} \\ - [n^2 - 5n + 2 \mp 2i\Omega M(6n-13)]b_{n-2} \mp 2i\Omega(3-n)b_{n-3} - 4c_{n-2}, \end{aligned} \quad (5.38)$$

$$2nM(n \mp 4i\Omega M)c_n = -2b_{n-1} - [n(n-3) \mp 8i\Omega M(n-1)]c_{n-1} \mp 2i\Omega(n-2)c_{n-2}. \quad (5.39)$$

For $n < 0$, a_n , b_n , and c_n are zero. For $n = 0$ and 1, Eqs. (5.37)–(5.39) allow a_0 , a_1 , and c_0 to be chosen freely. Other early coefficients in the sequences are

$$b_0 = \pm i\Omega R a_0, \quad (5.40)$$

$$b_1 = -\frac{R a_0}{2M^2} \mp i\Omega R a_1, \quad (5.41)$$

$$c_1 = \frac{c_0 \mp i\Omega R a_1}{M(1 \mp 4i\Omega M)}. \quad (5.42)$$

Similarly, for large r the exterior functions can be expanded as

$$A_{>}^\pm(r) = e^{\pm i\Omega r^*} \sum_{n=0}^{\infty} \hat{a}_n r^{-n}, \quad (5.43)$$

$$B_{>}^\pm(r) = e^{\pm i\Omega r^*} \sum_{n=0}^{\infty} \hat{b}_n r^{-n}, \quad (5.44)$$

$$C_{>}^\pm(r) = e^{\pm i\Omega r^*} \sum_{n=0}^{\infty} \hat{c}_n r^{-n}, \quad (5.45)$$

These forms ensure that the vector ξ_α satisfies outgoing-wave boundary conditions at infinity, another necessary condition to obtain a *retarded solution* to Eq. (5.13). Substitution into Eqs. (5.24)–(5.26) provides recurrence relations consisting of three coupled expressions for the \hat{a}_n , \hat{b}_n , and \hat{c}_n :

$$\begin{aligned} \pm 2i\Omega n \hat{a}_n &= [(n-2)(n+1) \pm 2i\Omega M(2n-3)]\hat{a}_{n-1} - 4M(n^2 - 3n + 1)\hat{a}_{n-2} \\ &\quad + 4M^2(n-2)(n-3)\hat{a}_{n-3} - 2MR^{-1}\hat{b}_{n-1}, \end{aligned} \quad (5.46)$$

$$\begin{aligned} \pm 2i\Omega n \hat{b}_n &= 2\Omega^2 R M \hat{a}_{n-1} + [n^2 - n - 4 \pm 2i\Omega M(2n-3)]\hat{b}_{n-1} - 4M(n^2 - 3n - 1)\hat{b}_{n-2} \\ &\quad + 4M^2(n^2 - 5n + 4)\hat{b}_{n-3} + 4\hat{c}_{n-1} - 16M\hat{c}_{n-2} + 16M^2\hat{c}_{n-3}, \end{aligned} \quad (5.47)$$

and

$$\pm 2i\Omega n \hat{c}_n = 2\hat{b}_{n-1} + (n-2)(n+1)\hat{c}_{n-1} - 2Mn(n-2)\hat{c}_{n-2}. \quad (5.48)$$

For $n < 0$, \hat{a}_n , \hat{b}_n , and \hat{c}_n are zero. Eqs. (5.46)–(5.48) allow \hat{a}_0 , \hat{b}_0 , and \hat{c}_0 to be chosen freely. Other early coefficients in the sequences are given by

$$\pm i\Omega \hat{a}_1 = -(1 \pm i\Omega M)\hat{a}_0 - M\hat{b}_0/R, \quad (5.49)$$

$$\pm i\Omega \hat{b}_1 = -M\Omega^2 R \hat{a}_0 - (2 \pm i\Omega M)b_0 + 2c_0, \quad (5.50)$$

$$\pm i\Omega \hat{c}_1 = b_0 - c_0. \quad (5.51)$$

The set of homogeneous solutions to Eqs. (5.24)–(5.26), inside and outside the orbit, forms a six dimensional linear vector space. The six amplitudes

$$a_0, \quad a_1, \quad c_0, \quad \hat{a}_0, \quad \hat{b}_0, \quad \hat{c}_0$$

determine one complete homogeneous solution and may be considered to be the “components” of any member of this vector space. The six amplitudes that generate the particular solution which satisfies the matching conditions of Eqs. (5.30)–(5.33) therefore identify the member of the vector space that corresponds to the desired gauge transformation.

Numerical integration of the ABC equations

The numerical integration of the homogeneous versions of Eqs. (5.24)–(5.26) is performed by first choosing starting points r_{\min} and r_{\max} which are close enough to their limiting values, respectively $2M$ and infinity, that the expansions and recursion relations (5.34)–(5.39) and (5.43)–(5.48) provide appropriate initial conditions for A , B , and C at machine accuracy with a reasonable number of terms in the sums (no more than 30 in our case). Also, the starting points are chosen to be sufficiently close to R that the resulting integration to R takes only a few seconds of machine time. Satisfying these two requirements simultaneously, both inside and outside the orbit, is not difficult in practice.

The integration routine requires six input parameters, the complex amplitudes a_0 , a_1 , c_0 , \hat{a}_0 , \hat{b}_0 , and \hat{c}_0 . These must be chosen so that the six jump conditions (5.30)–(5.33) are enforced. We have six algebraic equations for six unknowns. We pick a set of six linearly independent “basis solutions”, each of which has only one of the

$a_0 \dots \hat{c}_0$ equal to 1, all other amplitudes being zero. After integrating the basis solutions to R we collect the values of A , B , C , dA/dr , dB/dr and dC/dr , all evaluated at R , in a matrix

$$\mathbf{M} = \begin{pmatrix} -A_{1<} & -A_{2<} & -A_{3<} & A_{1>} & A_{2>} & A_{3>} \\ -B_{1<} & -B_{2<} & -B_{3<} & B_{1>} & B_{2>} & B_{3>} \\ -C_{1<} & -C_{2<} & -C_{3<} & C_{1>} & C_{2>} & C_{3>} \\ -A'_{1<} & -A'_{2<} & -A'_{3<} & A'_{1>} & A'_{2>} & A'_{3>} \\ -B'_{1<} & -B'_{2<} & -B'_{3<} & B'_{1>} & B'_{2>} & B'_{3>} \\ -C'_{1<} & -C'_{2<} & -C'_{3<} & C'_{1>} & C'_{2>} & C'_{3>} \end{pmatrix},$$

where we use an obvious notation; for example, $A'_{1<}$ is the value of dA/dr at $r = R$ for the first of the internal basis solutions. The required amplitudes of the six basis solutions form the unknown column vector \mathbf{x} , and the column vector \mathbf{j} contains the values of the discontinuities obtained from the jump conditions (5.30)–(5.33). After integrating the six basis solutions, we are left to solve the system of linear equations

$$\mathbf{M}\mathbf{x} = \mathbf{j} \quad (5.52)$$

for the desired amplitudes \mathbf{x} of our basis solutions; these then combine to give us the desired solution of Eqs. (5.24)–(5.26) with appropriate boundary conditions.

In our numerical work we use double-precision arithmetic and have adopted two different ODE integration routines from Chapter 16 of *Numerical Recipes* [32], the Runge-Kutta and the Burlish-Stoer algorithms. Each of these routines contains an accuracy parameter. A comparison of the numerical results over a range of values of this parameter allows us to be certain that all digits quoted in Table I are significant. We tested the consistency of the integrations versus the expansions by numerically integrating over a wide range in r where the expansions give accurate values for A , B and C . The consistency of the expansion routines with the integration routines is strong evidence that coding errors have been eliminated. Furthermore, we have written two independent codes, one per author, and all results were obtained independently before they were compared with each other. The agreement was well within the numerical errors of each code. Our final results for $A_{<}^\pm(R)$,

TABLE I: Computed values for the internal functions $A_{<}^{\pm}(R)$, $B_{<}^{\pm}(R)$ and $C_{<}^{\pm}(R)$. The external values are obtained by applying the jump conditions: $\text{Re}[A_{>}^{\pm}] = \text{Re}[A_{<}^{\pm}] + R^2$ and the imaginary parts are identical (similar statements hold for B and C). The functions $A^{-}(R)$, $B^{-}(R)$, and $C^{-}(R)$ are obtained by complex conjugation. All digits provided are significant. Note that we have set $M \equiv 1$ in our computations.

R	$\text{Re}[A_{<}^+]$	$\text{Im}[A_{<}^+]$	$\text{Re}[B_{<}^+]$	$\text{Im}[B_{<}^+]$	$\text{Re}[C_{<}^+]$	$\text{Im}[C_{<}^+]$
6	-39.427067	-0.68518043	-28.037347	3.1558616	-26.185013	3.8814154
7	-52.930571	-0.68313011	-38.997736	3.6225886	-37.151392	4.3105572
8	-68.389381	-0.66104724	-51.991922	4.0456669	-50.152019	4.6962809
9	-85.826868	-0.63522323	-67.000673	4.4373057	-65.166204	5.0547070
10	-105.25284	-0.61006840	-84.015966	4.8042522	-82.185790	5.3926745
11	-126.67197	-0.58677222	-103.03408	5.1508692	-101.20722	5.7139596
12	-150.08673	-0.56553614	-124.05317	5.4802524	-122.22887	6.0210382
13	-175.49853	-0.54624909	-147.07229	5.7947312	-145.24995	6.3157134
14	-202.90822	-0.52871248	-172.09095	6.0961288	-170.27013	6.5993856
15	-232.31636	-0.51271828	-199.10891	6.3859127	-197.28925	6.8731864
16	-263.72332	-0.49807505	-228.12606	6.6652898	-226.30729	7.1380554
17	-297.12936	-0.48461496	-259.14236	6.9352693	-257.32427	7.3947870
18	-332.53466	-0.47219375	-292.15781	7.1967063	-290.34024	7.6440626
19	-369.93935	-0.46068823	-327.17244	7.4503337	-325.35525	7.8864727
20	-409.34355	-0.44999342	-364.18629	7.6967858	-362.36939	8.1225343
25	-636.35940	-0.40594758	-579.24539	8.8386535	-577.42896	9.2219041
30	-913.37000	-0.37286764	-844.29131	9.8621836	-842.47459	10.213754
35	-1240.3777	-0.34683600	-1159.3279	10.797069	-1157.5106	11.123814
40	-1617.3835	-0.32564845	-1524.3578	11.662460	-1522.5399	11.969054
45	-2044.3881	-0.30796022	-1939.3826	12.471537	-1937.5641	12.761341
50	-2521.3918	-0.29289834	-2404.4036	13.233822	-2402.5846	13.509349
55	-3048.3949	-0.27986801	-2919.4216	13.956447	-2917.6021	14.219636
60	-3625.3975	-0.26844793	-3484.4373	14.644917	-3482.6173	14.897304
65	-4252.3997	-0.25833015	-4099.4510	15.303582	-4097.6306	15.546405
70	-4929.4016	-0.24928356	-4764.4632	15.935949	-4762.6423	16.170225
75	-5656.4033	-0.24113082	-5479.4740	16.544892	-5477.6528	16.771470
80	-6433.4048	-0.23373328	-6244.4838	17.132801	-6242.6622	17.352398
85	-7260.4061	-0.22698069	-7059.4926	17.701689	-7057.6707	17.914916
90	-8137.4073	-0.22078417	-7924.5006	18.253269	-7922.6784	18.460652
95	-9064.4083	-0.21507114	-8839.5079	18.789014	-8837.6855	18.991011
100	-10041.409	-0.20978162	-9804.5146	19.310200	-9802.6919	19.507212

$B_{<}^{\pm}(R)$ and $C_{<}^{\pm}(R)$ are listed in Table I for selected values of R . Results for $A_{<}^{-}(R)$, $B_{<}^{-}(R)$ and $C_{<}^{-}(R)$ are obtained by complex conjugation. Results for $A_{>}^{\pm}(R)$,

$B_{>}^{\pm}(R)$ and $C_{>}^{\pm}(R)$ are obtained from the jump conditions of Eq. (5.30).

Calculation of the self-acceleration

Substitution of Eqs. (5.27)–(5.29) into Eqs. (5.21)–(5.23), these into Eqs. (5.19), (5.20), and finally, these into Eq. (5.12) yields

$$h_{\alpha\beta} = h_{\alpha\beta}^s - \left([\xi_{\alpha}]r_{,\beta} + r_{,\alpha}[\xi_{\beta}] \right) \delta(r - R)$$

$$\begin{aligned} & - \left(\xi_{\alpha;\beta}^< + \xi_{\beta;\alpha}^< \right) \Theta(R - r) \\ & - \left(\xi_{\alpha;\beta}^> + \xi_{\beta;\alpha}^> \right) \Theta(r - R) \end{aligned}$$

in an obvious notation; for example $\xi_{\alpha}^<$ is the internal ($r < R$) solution to Eq. (5.13), constructed from $A_{<}(r)$, $B_{<}(r)$, and $C_{<}(r)$. The first three terms on the right-hand side appear to be singular, but it is easy to check

that by virtue of Eqs. (5.8)–(5.11) and (5.30), the factors multiplying $\delta(r - R)$ are all zero. We therefore have

$$h_{\alpha\beta} = -\left(\xi_{\alpha;\beta}^< + \xi_{\beta;\alpha}^<\right)\Theta(R - r) - \left(\xi_{\alpha;\beta}^> + \xi_{\beta;\alpha}^>\right)\Theta(r - R). \quad (5.53)$$

The jump conditions (5.31)–(5.33) also enforce

$$[\xi_{\alpha;\beta} + \xi_{\beta;\alpha}] = 0,$$

and we see that in the Lorenz gauge, the metric perturbation is continuous at $r = R$. Equation (5.53) also reveals that the internal ($r < R$) and external ($r > R$) forms of $h_{\alpha\beta}$ are obtained by a pure gauge transformation. The internal and external transformations, however, are distinct, and the perturbation is not globally pure gauge.

Differentiation of $h_{\alpha\beta}$ gives

$$\begin{aligned} h_{\alpha\beta;\gamma} &= -[\xi_{\alpha;\beta} + \xi_{\beta;\alpha}]r_{,\gamma}\delta(r - R) \\ &\quad - \left(\xi_{\alpha;\beta\gamma}^< + \xi_{\beta;\alpha\gamma}^<\right)\Theta(R - r) \\ &\quad - \left(\xi_{\alpha;\beta\gamma}^> + \xi_{\beta;\alpha\gamma}^>\right)\Theta(r - R). \end{aligned}$$

Once more the singular terms vanish and we end up with the nonsingular (but discontinuous) tensor

$$\begin{aligned} h_{\alpha\beta;\gamma} &= -\left(\xi_{\alpha;\beta\gamma}^< + \xi_{\beta;\alpha\gamma}^<\right)\Theta(R - r) \\ &\quad - \left(\xi_{\alpha;\beta\gamma}^> + \xi_{\beta;\alpha\gamma}^>\right)\Theta(r - R). \end{aligned} \quad (5.54)$$

This can now be substituted into Eq. (1.1) to obtain the $l = 1$, even-parity contribution to the self-acceleration; the calculation also involves the particle's velocity vector, $u^\mu = (1 - 3M/R)^{-1/2}[1, 0, 0, \Omega]$. In the notation of Eq. (1.3), we have

$$a[l = 1; \text{even}] = -3m\tilde{E}\frac{R - 2M}{R^4(R - 3M)}\text{Re}[B^+(R)], \quad (5.55)$$

which must be evaluated on either side of $r = R$. To arrive at Eq. (5.55) we have used the property that $B^-(R)$ is the complex conjugate of $B^+(R)$, so that $B^+(R) + B^-(R) = 2\text{Re}[B^+(R)]$. That the acceleration vector depends only on the radial component of ξ_α is a consequence of the facts that the acceleration is pure gauge (in the sense given above) and that the motion is circular. The curves displayed in Figs. 1 and 2 were obtained by substituting the numerical results of Table I into Eq. (5.55).

The $l = 1$, even-parity contribution to the self-acceleration takes different values depending on whether $B^+(R)$ is evaluated from inside or outside the orbit. By virtue of Eq. (5.30), its jump across the orbit is given by

$$[a[l = 1; \text{even}]] = -3m\tilde{E}\frac{R - 2M}{R^2(R - 3M)}. \quad (5.56)$$

When $R \gg M$, this agrees with the Newtonian result of Eq. (2.18).

Self-acceleration in the post-Newtonian limit

While we have not been able to find exact analytic solutions to Eqs. (5.24)–(5.26), it is possible to make some progress by linearizing the equations with respect to M . Solutions to these equations are then post-Newtonian approximations to the exact, numerically obtained solutions. We now set out to obtain these approximations, and to compare them with the numerical results.

After linearization — recall that $\Omega^2 = M/R^3$ is linear in M — the homogeneous equations become

$$\frac{d^2 A}{dr^2} + \left(\Omega^2 - \frac{2}{r^2} - \frac{4M}{r^3}\right)A - \frac{2M/R}{r^2}B = 0, \quad (5.57)$$

$$\frac{d^2 B}{dr^2} + \left(\Omega^2 - \frac{4}{r^2} - \frac{4M}{r^3}\right)B + \frac{4}{r^2}C = 0, \quad (5.58)$$

and

$$\frac{d^2 C}{dr^2} + \frac{2M}{r^2}\frac{dC}{dr} + \left(\Omega^2 - \frac{2}{r^2} - \frac{4M}{r^3}\right)C + \left(\frac{2}{r^2} + \frac{4M}{r^3}\right)B = 0, \quad (5.59)$$

where we have omitted the \pm labels for ease of notation. The jump conditions reduce to $[A] = [B] = [C] = R^2$ and

$$\begin{aligned} \left[\frac{dA}{dr}\right] &= 2R\left(1 + \frac{2M}{R}\right), \\ \left[\frac{dB}{dr}\right] &= R\left(1 + \frac{M}{R}\right), \\ \left[\frac{dC}{dr}\right] &= R\left(1 - \frac{2M}{R}\right). \end{aligned} \quad (5.60)$$

We notice that the equations for B and C decouple from the equation for A . In the sequel we will construct solutions to the B and C equations, and leave $A(r)$ undetermined; for the purposes of calculating the self-acceleration, only $B(r)$ is required. Our solutions will satisfy outgoing-wave boundary conditions at $r \rightarrow \infty$, so that in the following, $B(r) \equiv B^+(r)$ and $C(r) \equiv C^+(r)$.

To decouple the B and C equations we introduce the new dependent variables $\psi_- = [B - (1 - M/r)C]/R^2$ and $\psi_+ = [\frac{1}{3}B + \frac{2}{3}(1 - M/r)C]/R^2$, such that

$$B = R^2\left(\psi_+ + \frac{2}{3}\psi_-\right) \quad (5.61)$$

and

$$C = R^2\left(1 + \frac{M}{r}\right)\left(\psi_+ - \frac{1}{3}\psi_-\right). \quad (5.62)$$

Away from $r = R$, these functions satisfy the differential equations

$$\psi_-'' + \left(1 - \frac{6}{z^2} - \frac{6v^3}{z^3}\right)\psi_- = 0 \quad (5.63)$$

and

$$\psi''_+ + \psi_+ = 0, \quad (5.64)$$

where we have introduced the rescaled independent variable $z = \Omega r$ and the small quantity $v^3 = M\Omega = (M/R)^{3/2}$; a prime indicates differentiation with respect to z . In terms of the new variables, the jump conditions become

$$[\psi_-] = v^2, \quad [\psi'_-] = 3v \quad (5.65)$$

and

$$[\psi_+] = 1 - \frac{2}{3}v^2, \quad [\psi'_+] = \frac{1}{v}(1 - v^2); \quad (5.66)$$

matching is carried out at $z = v$.

To find solutions to Eq. (5.63) we use the fact that v is small and write

$$\psi_- = \psi_0 + v^3\psi_1 + O(v^6). \quad (5.67)$$

Substitution into Eq. (5.63) yields an equation for ψ_0 ,

$$\psi''_0 + \left(1 - \frac{6}{z^2}\right)\psi_0 = 0, \quad (5.68)$$

and another equation for ψ_1 ,

$$\psi''_1 + \left(1 - \frac{6}{z^2}\right)\psi_1 = \frac{6}{z^3}\psi_0. \quad (5.69)$$

We first solve these equations in the domain $z < v$. Among all possible solutions to Eq. (5.63), we choose one which does not diverge in the limit $z \rightarrow 0$. While this condition seems appropriate for our purposes, it is important to understand that we cannot fully justify it here: this choice must be introduced as an additional assumption. The reason is as follows: Linearization of the equations with respect to M implies that Eqs. (5.63) and (5.64) apply only in the domain $r \gg M$, or $z \gg v^3$, and this restriction prevents us from imposing a proper incoming-wave condition at the horizon ($r = 2M$, or $z = 2v^3$). We must therefore identify a suitable replacement for this boundary condition, in the form of an asymptotic condition holding when z is restricted by $v^3 \ll z \ll v$. Previous experience [33, 34] with solving the Regge-Wheeler equation [25] in the low-frequency limit ($M\Omega = v^3 \ll 1$) suggests that an appropriate substitution is a regularity condition in the formal limit $z \rightarrow 0$. This is the choice we make here, without confirmation that this conclusion applies to the system (5.24)–(5.26).

A regular solution to Eq. (5.68) is $\psi_0(z) = zj_2(z)$, or

$$\psi_0^<(z) = \left(\frac{3}{z^2} - 1\right) \sin z - \frac{3}{z} \cos z, \quad (5.70)$$

where $j_2(z)$ is a spherical Bessel function. Substituting Eq. (5.70) into Eq. (5.69) and integrating returns a linear superposition of $zj_2(z)$, $zn_2(z)$, and $\psi_p(z)$, a particular

solution to the differential equation. The term involving $zj_2(z)$ can be discarded, as it simply renormalizes the zeroth-order solution of Eq. (5.70). The coefficient in front of $zn_2(z)$ must then be chosen so as to yield a regular solution. This gives

$$\psi_1^<(z) = -\frac{3}{2}\left(\frac{1}{z} - \frac{2}{z^3}\right) \sin z + \frac{1}{2}\left(1 - \frac{6}{z^2}\right) \cos z, \quad (5.71)$$

and the complete interior solution to Eq. (5.63) is

$$\psi_-^<(z) = a[\psi_0^< + v^3\psi_1^< + O(v^6)]. \quad (5.72)$$

The amplitude a will be determined by matching.

We next turn to the domain $z > v$ and construct an exterior solution to Eq. (5.63); this will be required to satisfy an outgoing-wave condition as $z \rightarrow \infty$. The procedure is largely the same as for the interior solution, but is simplified by the fact that the outer boundary is part of the domain $z \gg v^3$. An outgoing-wave solution to Eq. (5.68) is $\psi_0(z) = -izh_2^{(1)}(z)$, or

$$\psi_0^>(z) = \left(1 + \frac{3i}{z} - \frac{3}{z^2}\right) e^{iz}. \quad (5.73)$$

Substituting this into Eq. (5.69) and integrating returns a linear superposition of $zh_2^{(1)}(z)$, $zh_2^{(2)}(z)$, and $\psi_p(z)$, a particular solution to the differential equation. As before the term involving $zh_2^{(1)}(z)$ can be discarded, and the term involving $zh_2^{(2)}(z)$ must also be eliminated because it represents an incoming wave. We are left with the particular solution,

$$\psi_1^>(z) = \frac{3i}{2}\left(\frac{1}{z^2} + \frac{2i}{z^3}\right) e^{iz}. \quad (5.74)$$

The complete exterior solution to Eq. (5.63) is then

$$\psi_-^>(z) = b[\psi_0^> + v^3\psi_1^> + O(v^6)], \quad (5.75)$$

and the amplitude b will be determined by matching.

The constants a and b are determined by inserting Eqs. (5.72) and (5.75) into the jump conditions of Eq. (5.65). The results are moderately complicated, and we shall not display them here. The expressions, however, simplify once we take into account the fact that v is small. At the matching point we find

$$\psi_-^<(v) = -v^2 + O(v^4), \quad \psi_-^>(v) = O(v^4). \quad (5.76)$$

In the interior domain ($z < v$) we can take advantage of the fact that z is formally of order v to derive

$$\psi_-^<(z) = -(z/v)^3 v^2 + O(v^4). \quad (5.77)$$

We now proceed with finding interior and exterior solutions to Eq. (5.64). This is a much simpler task, but as we shall see, our solutions will not be fully determined. For an interior solution we write

$$\psi_+^<(z) = \alpha(\sin z - \beta v^3 \cos z), \quad (5.78)$$

where α and β are constants; the scaling of the cosine term with v^3 is introduced for convenience, in anticipation of later results. We note that this solution is regular in the formal limit $z \rightarrow 0$, in agreement with the discussion given previously, and that it involves two undetermined constants. For an exterior solution we choose

$$\psi_+^>(z) = \gamma e^{iz}, \quad (5.79)$$

where γ is another constant. Substitution of Eqs. (5.78) and (5.79) into the jump conditions of Eq. (5.66) allows us to determine α and γ , but β is left over as a free parameter. Once more the resulting expressions are too complicated to be displayed, but they simplify for $v \ll 1$. At the matching point we find

$$\begin{aligned} \psi_+^<(v) &= -1 + \left(\beta + \frac{2}{3}\right)v^2 + O(iv^3, v^4), \\ \psi_+^>(v) &= \beta v^2 + O(iv^3, v^4), \end{aligned} \quad (5.80)$$

and in the interior domain we have

$$\psi_+^<(z) = -(z/v) + \left[\beta + \frac{1}{2}(z/v) + \frac{1}{6}(z/v)^3\right]v^2 + O(iv^3, v^4). \quad (5.81)$$

Substitution of Eqs. (5.76) and (5.80) into Eqs. (5.61) and (5.62) yields

$$B_<(R) = C_<(R) = -R^2 \left[1 - \beta \frac{M}{R} + O(iM\Omega, M^2/R^2)\right], \quad (5.82)$$

as well as $B_>(R) = C_>(R) = \beta MR + \dots$. According to these results, Eq. (5.55) becomes

$$a_<[l = 1; \text{even}] \simeq \frac{3m\tilde{E}}{R^2} \left[1 - (\beta - 1)\frac{M}{R}\right] \quad (5.83)$$

and

$$a_>[l = 1; \text{even}] \simeq -\frac{3m\tilde{E}}{R^2} \frac{\beta M}{R}. \quad (5.84)$$

This is compatible with the Newtonian results presented in Eq. (2.17).

Substitution of Eqs. (5.77) and (5.81) into Eqs. (5.61) and (5.62) gives us expressions for the interior functions:

$$\begin{aligned} B_<(r) &= -R^2 \left\{ (r/R) - \left[\beta + \frac{1}{2}(r/R) - \frac{1}{2}(r/R)^3 \right] \frac{M}{R} \right. \\ &\quad \left. + O(iM\Omega, M^2/R^2) \right\} \end{aligned} \quad (5.85)$$

and

$$\begin{aligned} C_<(r) &= -R^2 \left\{ (r/R) - \left[\beta - 1 + \frac{1}{2}(r/R) \right. \right. \\ &\quad \left. \left. + \frac{1}{2}(r/R)^3 \right] \frac{M}{R} + O(iM\Omega, M^2/R^2) \right\}. \end{aligned} \quad (5.86)$$

These solutions are parameterized by β , which cannot be determined here because of our lack of control over the behavior of the solutions near $r = 2M$.

We can, however, estimate the value of β by fitting Eq. (5.82) to our numerical results. We proceed as follows. First, we fit the expression $1 - \beta M/R + \beta'_B(M/R)^2$ to our numerical values for $B_<(R)/(-R^2)$ in the interval $20 \leq R/M \leq 100$; this yields $\beta = 1.9936 \pm 0.0006$ and $\beta'_B = 4.07 \pm 0.02$. Second, we fit the expression $1 - \beta M/R + \beta'_C(M/R)^2$ to our numerical values for $C_<(R)/(-R^2)$ restricted to the same interval; this yields $\beta = 1.9936 \pm 0.0006$ and $\beta'_C = 2.26 \pm 0.02$. Third, we fit the expression $\beta + \beta''_B(M/R)$ to our numerical values for $B_>(R)/(MR)$; this yields $\beta = 1.9951 \pm 0.0004$ and $\beta''_B = -4.12 \pm 0.02$. Finally, we fit the expression $\beta + \beta''_C(M/R)$ to our numerical values for $C_>(R)/(MR)$; this yields $\beta = 1.9952 \pm 0.0004$ and $\beta''_C = -2.31 \pm 0.02$. We notice an excellent consistency among our estimates of β , and we conclude that according to our numerical results,

$$\beta = 1.994 \pm 0.001.$$

It is probable that the actual value is $\beta = 2$, and that the slight discrepancy results from a failure to include additional terms in the expansions in powers of M/R . The two-parameter fits presented here were obtained with a nonlinear least-squares Marquardt-Levenberg algorithm, as implemented in the software **gnuplot**.

The quality of the fits can be judged by comparing the numerically-obtained functions $B_<(r)$ and $C_<(r)$ with the post-Newtonian approximations of Eqs. (5.85) and (5.86), in which we substitute $\beta = 2$. We present this comparison in Fig. 3 for $R = 25M$. We see that the analytic expressions are very accurate for all values of $r < R$ except near $r = 2M$.

VI. DISCUSSION

Using the tensor harmonic decomposition of Regge and Wheeler [25], Zerilli [26] and many others have studied the metric perturbations resulting from the geodesic motion of a small mass in the background geometry of a Schwarzschild black hole. Most of the attention was devoted to the radiating modes, those with $l \geq 2$, and their analysis typically involves numerical work.

Much less attention has been garnered by the non-radiating, $l = 0$ and $l = 1$ modes. In fact, the vacuum $l = 1$ even-parity metric perturbations were shown to be *just gauge* by Zerilli. This mode contains no gravitational radiation, and is usually ignored in analyses involving gravitational perturbations of black holes. Nonetheless, this very mode plays an important role in self-force calculations: only the dipole mode has a Newtonian-order contribution to the self force, as we have shown in Sec. II. The dipole metric perturbations cannot be ignored.

Zerilli found analytic expressions for the $l = 0$ and $l = 1$ metric perturbations in a convenient gauge. The

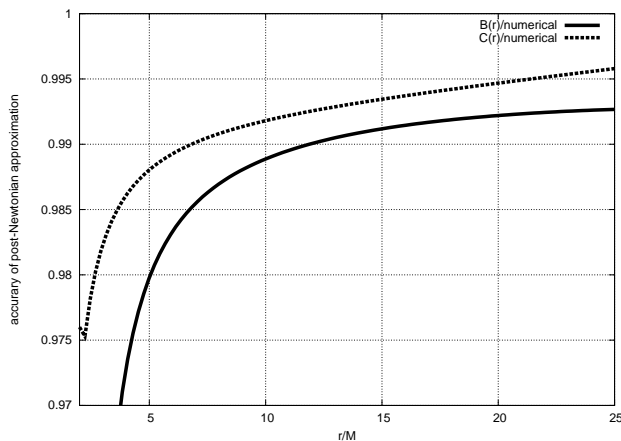


FIG. 3: Accuracy of the post-Newtonian expressions for the functions $B_{<}(r)$ and $C_{<}(r)$, for $R = 25M$. The solid curve is a plot of $\text{Re}[B_{<}(r)]$, as given by Eq. (5.85), divided by the numerical results listed in Table I. The dashed curve is a plot of $\text{Re}[C_{<}(r)]$, as given by Eq. (5.86), divided by the numerical results listed in Table I. In both cases we have set $\beta = 2$. The error is estimated to be of order $(M/R)^2 \simeq 0.002$. The plots reveal that this estimate is accurate for all values of r except near $r = 2M$.

Lorenz gauge, however, with its hyperbolic wave operator, is preferred for self-force calculations. We, as well as Nakano, Sago, and Sasaki [31], have found analytic expressions for the $l = 0$ and odd-parity $l = 1$ cases. Our analysis of the even-parity $l = 1$ case is mostly numerical, but our procedure is robust and easy to implement.

While the Lorenz-gauge treatment of these non-radiating modes is now in hand, this analysis is but a small part of a complete computation of the regularized self-acceleration, a program that was outlined in Sec. I. And the ultimate goal of incorporating the equations of motion, with their corrections of order m/M , into a wave-generation formalism to obtain accurate gravitational-wave templates, remains elusive.

For example, the conservative forces discussed in this paper affect the trajectory of the small mass at order m/M . But the description of this effect inherently depends upon the choice of gauge. While the actual observation of a gravitational-wave signal at a large distance from the system is a gauge-independent measurement, the details of the conversion from the self-force, as measured in the Lorentz gauge, to the m/M corrections to the wave forms, which are gauge invariant, are not yet known.

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APPENDIX A: MONOPOLE GAUGE TRANSFORMATION WITHIN THE LORENZ GAUGE

In section III we calculated the self-acceleration from the monopole metric perturbation in the Lorenz gauge and claimed that this result was unique. Nevertheless, our results differ from the post-Newtonian results of Nakano, Sago, and Sasaki (NSS) [31], who also work in the Lorenz gauge. In this Appendix we outline a possible cause for the discrepancy.

Both groups begin with differing solutions in the Zerilli gauge and then find differing gauge transformations to the Lorenz gauge, with resulting metric perturbations that yield differing accelerations. We ask: do our results differ from NSS by a gauge transformation from one Lorenz gauge (ours) to another Lorenz gauge (theirs)? An affirmative answer would invalidate our statement that our choice of Lorenz gauge is unique. We shall instead argue that while our results are indeed related by a gauge transformation, this transformation takes our Lorenz gauge into another Lorenz gauge that fails to be regular on the event horizon. We are therefore correct in stating that our gauge choice is unique, because the gauge employed by NSS, while appropriate for a post-Newtonian treatment, does not have a proper relativistic generalization.

Before we investigate this matter, we note that our expression for the metric perturbation in the Zerilli gauge has

$$h_{tt}^Z = \frac{2m\tilde{E}}{r} \left(1 - \frac{r-2M}{R-2M}\right) \Theta(r-R), \quad (\text{A1})$$

which is zero inside the orbit; this property simplifies our implementation of the boundary conditions at the horizon. Nakano, Sago, and Sasaki, their Eq. (E5), have instead

$$h_{tt}^{\text{NSS}} = \frac{2m\tilde{E}}{r} \left[\frac{r-2M}{R-2M} \Theta(R-r) + \Theta(r-R) \right], \quad (\text{A2})$$

which inside the orbit is very similar to the Newtonian result of Eq. (2.12). The difference

$$h_{tt}^{\text{NSS}} - h_{tt}^{\text{Z}} = \frac{2m\tilde{E}}{r} \frac{r - 2M}{R - 2M} \quad (\text{A3})$$

is a gauge transformation generated by $\xi^\alpha = [\alpha t, 0, 0, 0]$ with $\alpha = m\tilde{E}/(R - 2M)$, so that

$$-2\xi_{t;t} = h_{tt}^{\text{NSS}} - h_{tt}^{\text{Z}} = \frac{2m\tilde{E}}{r} \frac{r - 2M}{R - 2M}. \quad (\text{A4})$$

All other components of $\xi_{(\alpha;\beta)}$ are zero.

Now we seek an answer to our earlier questions. We assume that the $l = 0$ metric perturbation has already been obtained in one Lorenz gauge (ours, the results appearing in Sec. III), and we ask whether it is possible to take it to another Lorenz gauge (which we imagine to be a relativistic generalization of the choice made by NSS). We consider the most general $l = 0$ gauge vector

$$j^\alpha = [\alpha t, j(r), 0, 0] \quad (\text{A5})$$

that keeps the metric perturbation static, where α is now an arbitrary constant. The gauge transformation generated by this vector produces a shift in the metric perturbation given by $\Delta h_{\alpha\beta} = -2j_{(\alpha;\beta)}$. For the shifted perturbation to satisfy the Lorenz gauge condition, the gauge vector must satisfy the wave equation $\square j^\alpha = 0$. This gives

$$j'' + \frac{2}{r}j' - \frac{2}{r^2}j = \frac{2\alpha M}{r^2 f}, \quad (\text{A6})$$

where $f = 1 - 2M/r$. The general solution to this wave equation is

$$j = c_1 r + c_2 \frac{M^3}{r^2} + \frac{\alpha}{9r^2} \left[3r^3 \ln(1 - 2M/r) - 3Mr^2 - 12M^2 r + 44M^3 - 24M^3 \ln(r/2M - 1) \right], \quad (\text{A7})$$

where c_1 and c_2 are arbitrary, dimensionless constants. Notice that the complicated function within the square brackets is closely related to the function $\Gamma(r)$ defined in Eq. (3.10), namely $[\] = -9Mr^2 f \Gamma(r)$. The resulting non-zero shifts in the metric perturbation are

$$\begin{aligned} \Delta h_{tt} &= 2\alpha f + \frac{2M}{r^2} j(r), \\ \Delta h_{rr} &= -\frac{2}{f} \frac{dj(r)}{dr} + \frac{2M}{r^2 f^2} j(r), \\ \Delta h_{\theta\theta} &= \sin^2 \theta \Delta h_{\phi\phi} = -2r j(r). \end{aligned} \quad (\text{A8})$$

We must now find constants c_1 , c_2 , and α which yield a regular metric perturbation everywhere, including on the event horizon and at infinity.

Behavior at the event horizon is easily examined in Eddington-Finkelstein coordinates

$$\mathcal{V} = t + r + 2M \ln(r/2M - 1), \quad \mathcal{R} = r. \quad (\text{A9})$$

The coordinates \mathcal{V} and \mathcal{R} are well defined everywhere in the vicinity of the future horizon, which is located at $\mathcal{R} = 2M$. One component of j^α in Eddington-Finkelstein coordinates is

$$\begin{aligned} j^\mathcal{V} &= j^t + j^r/f \\ &= \alpha [\mathcal{V} - \mathcal{R} - 2M \ln(\mathcal{R}/2M - 1)] + j(\mathcal{R}) \end{aligned} \quad (\text{A10})$$

With a substitution from Eq. (A7) it is seen that regularity of $j^\mathcal{V}$ at the future horizon requires that

$$2c_1 + \frac{1}{4}c_2 + \frac{2}{9}\alpha = 0, \quad (\text{A11})$$

and with this same condition $j^\mathcal{R} = O(f)$. We still need to check the regularity of $\Delta h_{\alpha\beta}$ in Eddington-Finkelstein coordinates,

$$\begin{aligned} \Delta h_{\mathcal{V}\mathcal{V}} &= \Delta h_{tt}, \\ \Delta h_{\mathcal{V}\mathcal{R}} &= \Delta h_{tt}/f, \\ \Delta h_{\mathcal{R}\mathcal{R}} &= \Delta h_{tt}/f^2 + \Delta h_{rr}. \end{aligned} \quad (\text{A12})$$

With the choice of constants in Eq. (A11), we see that

$$\Delta h_{tt} = \alpha f [1 + \ln(r/2M - 1)] + O(f^2), \quad (\text{A13})$$

and

$$\Delta h_{tt}/f^2 + \Delta h_{rr} = 2\alpha + O(f). \quad (\text{A14})$$

Thus $\Delta h_{\mathcal{V}\mathcal{V}}$ and $\Delta h_{\mathcal{R}\mathcal{R}}$ are regular on the future horizon, but $\Delta h_{\mathcal{V}\mathcal{R}}$ is singular if α is not zero. Further analysis shows that a choice of constants which does not satisfy Eq. (A11) only makes the shifts more singular. Examination of behavior on the past horizon leads to the same conclusion. The only condition on the constants that makes $\Delta h_{\alpha\beta}$ a regular tensor field on the horizon is $\alpha = 0$ and $c_1 = -c_2/8$; but $\Delta h_{\alpha\beta}$ is then ill behaved as $r \rightarrow \infty$. We conclude that there is no monopole gauge transformation that simultaneously preserves the Lorenz gauge condition and behaves properly on the event horizon and at infinity. This confirms that our claim was true: our choice of Lorenz gauge is indeed unique.

Nonetheless, if we set $c_1 = 0$ and make a gauge transformation with Eq. (A5), we find that the resulting change in the self-acceleration is completely due, at first post-Newtonian order, to $j^t = \alpha t$; the contribution from $j^r = j(r)$ appears at higher post-Newtonian order. With the value of α set to $m\tilde{E}/(R - 2M)$, the first post-Newtonian contribution to the self-acceleration is $-4mM/R^3$, and it completely accounts for the difference between our results and those of Nakano, Sago, and Sasaki [31], who present first post-Newtonian results in their Eq. (E19). From the discussion provided above, we conclude that our results are related by a gauge transformation, but that this transformation takes our Lorenz gauge into a Lorenz gauge that fails to be regular on the event horizon. In other words, the Lorenz gauge employed by NSS, while appropriate for their post-Newtonian treatment, does not have a proper relativistic generalization.

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